PHY489 Week 11:

Introduction to Quantum Chromodynamics

Quarks have flavour (u, d, s, b, c, t) and color (r, g, b).

Can represent colours with vectors:

$$c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^{(\text{red})}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^{(\text{blue})}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^{(\text{green})}$$

Thus a quark is described by a Dirac spinor and a color vector

$$u_{(s)}(p)\, c.$$

Gluons

→ Massless spin-1, specified in color basis (couples to itself)

$$\left\{\begin{array}{ll} |1\rangle = \frac{1}{\sqrt{2}}(r\bar{b} + b\bar{r}) & |5\rangle = -\frac{i}{\sqrt{2}}(r\bar{g} - g\bar{r}) \\ |2\rangle = \frac{-i}{\sqrt{2}}(r\bar{b} - b\bar{r}) & |6\rangle = \frac{1}{\sqrt{2}}(b\bar{g} + g\bar{b}) \\ |3\rangle = \frac{1}{\sqrt{2}}(r\bar{r} - b\bar{b}) & |7\rangle = -\frac{i}{\sqrt{2}}(b\bar{g} - g\bar{b}) \\ |4\rangle = \frac{1}{\sqrt{2}}(r\bar{g} + g\bar{r}) & |8\rangle = \frac{1}{\sqrt{6}}(r\bar{r} + b\bar{b} - 2g\bar{g}) \end{array}\right\}$$

→ Vector 'a' 8-component.

ie.

$$a = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv |1\rangle.$$

QCD is based on $SU(3)$, we introduce Gell-Mann λ-matrices (sim. to $SU(2)$ Pauli matrices) where are rotation generators:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

with Lie Algebra $[\lambda_a, \lambda_b] = 2i f^{abc} \lambda_c$

f^{abc} "Structure constants"

$f^{123} = 1$

$f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}$

$f^{458} = f^{678} = \frac{\sqrt{3}}{2}$.

Feynman Rules (QCD)

1. External Lines

Quark: $\begin{cases} \text{Incoming: } u^{(s)}(p)\, c \\ \text{Outgoing: } \bar{u}^{(s)}(p)\, c^{\dagger} \end{cases}$

Antiquark: $\begin{cases} \text{Incoming: } \bar{v}^{(s)}(p)\, c^{\dagger} \\ \text{Outgoing: } v^{(s)}(p)\, c \end{cases}$

Gluon: $\begin{cases} \text{Incoming } (\alpha, \mu) : \epsilon_\mu(p)\, a^{\alpha} \\ \text{outgoing } (\alpha, \mu) : \epsilon^*_\mu(p)\, a^{\alpha *} \end{cases}$

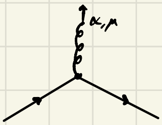
2. Propagators

Quarks/Antiquarks: $\dfrac{i(\not{q} + mc)}{q^2 - m^2c^2}$

Gluons (q; α,μ — β,ν): $\dfrac{-i g_{\mu\nu} \delta^{\alpha\beta}}{q^2}$

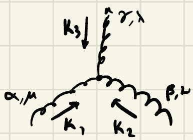
3. Vertices

Quark - Quark - Gluon:



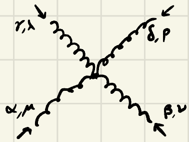
$$-\frac{ig_s}{2}\lambda^\alpha\gamma^\mu$$

Three - Gluon:



$$-g_s f^{\alpha\beta\gamma}\left[g_{\mu\nu}(k_1-k_2)_\lambda + g_{\nu\lambda}(k_2-k_3)_\mu + g_{\lambda\mu}(k_1-k_3)_\nu\right]$$

Four - Gluon:



$$-ig_s^2\left[f^{\alpha\beta\eta}f^{\gamma\delta\eta}\left[g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}\right] + f^{\alpha\delta\eta}f^{\beta\gamma\eta}\left[g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}\right] + f^{\alpha\gamma\eta}f^{\delta\beta\eta}\left[g_{\mu\rho}g_{\nu\lambda}-g_{\mu\nu}g_{\lambda\rho}\right]\right]$$

Coupling Constants: in QED, as internal momentum q increases, we get larger coupling constants.

In QCD, as q increases, the coupling constants decrease in higher-order diagrams.

fine structure $\frac{1}{137}$

→ Effective charge $$\alpha_E(q^2) = \frac{\alpha_E(0)}{1-(\alpha_E(0)/3\pi)\log(|q|^2/m_e^2)} \qquad q^2 \gg (m_e c)^2$$

"Running" Coupling constants
(Renormalization)

$$\alpha_s(q^2) = \frac{\alpha_s(\mu^2)}{1+\left[\alpha_s(\mu^2)/12\pi\right](11n-2f)\log(|q|^2/\mu^2)} \qquad q^2 \gg \mu^2$$

n: # of colours
f: # of flavours

μ: any constant (parametrizes coupling)

$$\log\Lambda^2 = \log\mu^2 - 12\pi/\left[(11n-2f)\alpha_s(\mu^2)\right]$$

$\Rightarrow \Lambda_{QCD} \doteq 220$ MeV where $\alpha_s \approx 1$.

Hadronization: only colour singlet states are observed.

- "Quarks free at high energies" ≡ QCD coupling decreases as q^2 increases.

(QCD well below Λ_{QCD})

BARYON WAVEFUNCTIONS

- Spin state $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = 0, 1 \otimes \frac{1}{2} = \frac{1}{2}, \frac{3}{2}$

Omit relative orbital angular momentum.

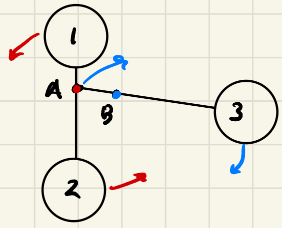
- uds-quark model: $3^3 = 27$ qqq flavour combinations

Qu: symmetries?

Qu: color? Also 3^3 combinations to consider.

Angular momentum:

only need 2 ℓ values ℓ, ℓ' to specify orbital L, but remain in ground state.



- Total wavefunction must be antisymmetric under the exchange of two particles (true for any fermion)

See pauli / Dirac spin statistics $\{\psi_1, \psi_2\} = 0$
(spinors)

The total wavefunction must then be specified by

$$\psi_{tot} = \psi(\text{spatial})\, \psi(\text{spin})\, \psi(\text{flavour})\, \psi(\text{colour})$$

and must be odd under the exchange of any two quarks.

Ground state is always symmetric

- For spin states $(\frac{1}{2}, \frac{3}{2})$ there are a total of 8 combinations (2^3)

$\uparrow\uparrow\uparrow$ — (symmetric)

$\uparrow\uparrow\downarrow$ $\uparrow\downarrow\uparrow$ $\downarrow\uparrow\uparrow$

$\downarrow\downarrow\uparrow$ $\downarrow\uparrow\downarrow$ $\uparrow\downarrow\downarrow$

$\downarrow\downarrow\downarrow$ — (symmetric)

Hence, for total angular momentum eigenstates, we have

$$|\tfrac{3}{2}\ \tfrac{3}{2}\rangle = |{+}{+}{+}\rangle$$

$$|\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\left(|{+}{+}{-}\rangle + |{+}{-}{+}\rangle + |{-}{+}{+}\rangle\right)$$

$$\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left(|{+}{-}{-}\rangle + |{-}{+}{-}\rangle + |{-}{-}{+}\rangle\right)$$

$$\left|\tfrac{3}{2}\,-\tfrac{3}{2}\right\rangle = |{-}{-}{-}\rangle .$$

There are also mixed symmetry states:

$$\left.\begin{aligned}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle_{12} &= \frac{1}{\sqrt{2}}\left(|{+}{-}\rangle - |{-}{+}\rangle\right)\otimes|{+}\rangle \\ \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_{12} &= \frac{1}{\sqrt{2}}\left(|{+}{-}\rangle - |{-}{+}\rangle\right)\otimes|{-}\rangle\end{aligned}\right\}\text{ antisymmetric in } 1,2$$

$$\left.\begin{aligned}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle_{2,3} &= \frac{1}{\sqrt{2}}|{+}\rangle\otimes\left(|{+}{-}\rangle - |{-}{+}\rangle\right) \\ \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_{2,3} &= \frac{1}{\sqrt{2}}|{-}\rangle\otimes\left(|{+}{-}\rangle - |{-}{+}\rangle\right)\end{aligned}\right\}\text{ antisymmetric in } 2,3$$

How do we add the color and flavor Hilbert spaces?

3^3 combinations of each, but they may also be written as combinations with pure/mixed symmetry under particle exchange.

- Pure symmetry states:

uuu, sss, ddd (3)

$\frac{1}{\sqrt{3}}(uus + usu + suu)$ (7 – also for s,d,u, ...)

One Fully antisymmetric state

$$\frac{1}{\sqrt{6}}\left(uds - usd + dsu - dus + sud - sdu\right)$$

16 mixed symmetry states

↳ 8 (anti-symmetry between $1 \leftrightarrow 2$)

↳ 8 (anti-symmetry between $2 \leftrightarrow 3$)

And the same argument applies to colour states

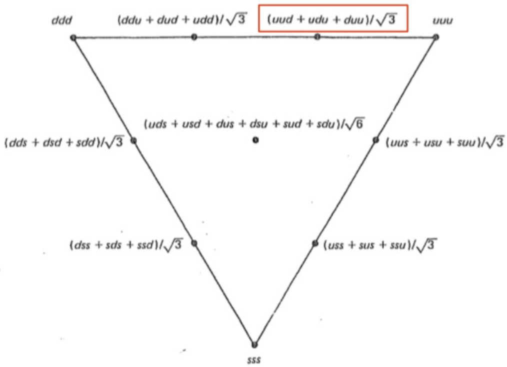
<u>Quarks only exist in colourless objects</u>

All <u>free</u> particles belong to a colour singlet

$$\psi(\text{colour}) = \frac{1}{\sqrt{6}}\left(rgb - rbg + gbr - grb + brg - bgr\right)$$

(Ground state baryons must therefore be symmetric)

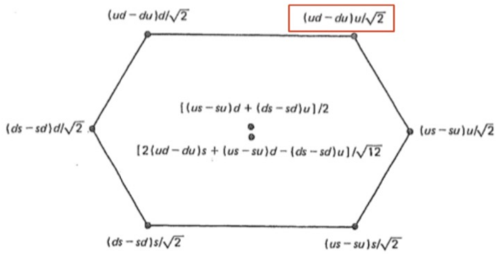
Flavour multiplets (uds quark model)



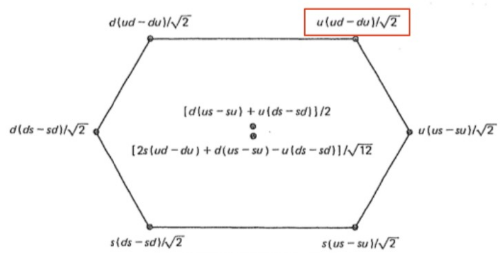
ψ_s: Completely symmetric states

(uds − usd + dsu − dus + sud − sdu)/√6

ψ_A: Completely antisymmetric state



ψ_{12}: Antisymmetric in 1 and 2



ψ_{23}: Antisymmetric in 2 and 3